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PROPOSITION IN TRANSVERSALS.

BY PROF. E. W. HYDE, CINCINNATI UNIVERSITY, CIN., OHIO.

LET any transversal OY be drawn to the triangle ABC , cutting its sides in P_1, P_2 and P_3 : from any point, O , on the transversal, lay off distances OP_1, OP_2 and OP_3 such that $OP_1 \times OP_1 = OP_2 \times OP_2 = OP_3 \times OP_3 = \overline{OD}^2 = c^2$, say: also lay off distances OQ_1, OQ_2 and OQ_3 in the opposite direction from O , such that $OP_1 \times OQ_1 = OP_2 \times OQ_2 = OP_3 \times OQ_3 = c^2$: then $P'A, P''B$ and $P'''C$ will meet in a point O' , and $Q'A, Q''B$ and $Q'''C$ will also meet in a point, O'' .

The distance OD may be of any length.

Through O , draw OX perpendicular to OY and take these lines for the axes of reference. The equations of BC, CA and AB will be

$$y - m_1 x - b_1, \quad (1)$$

$$y - m_2 x - b_2, \quad (2)$$

$$y = m_3 x + b_3, \quad (3)$$

so that $b_1 = OP_1$, $b_2 = OP_2$ and $b_3 = OP_3$. The equations of lines through A, B and C will be

$$y - m_2 x - b_2 = k_1(y - m_3 x - b_3), \quad (4)$$

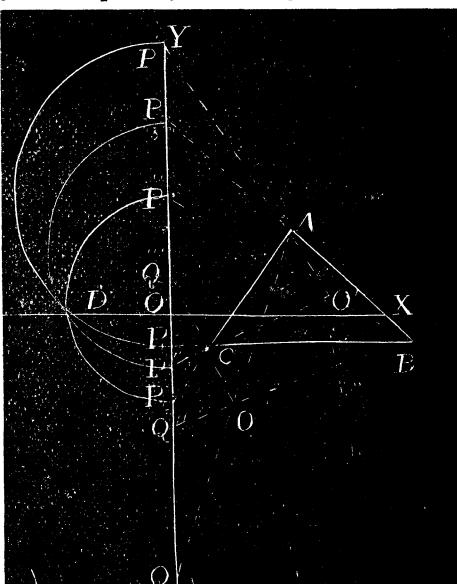
$$y - m_3 x - b_3 = k_2(y - m_1 x - b_1), \quad (5)$$

$$y - m_1 x - b_1 = k_3(y - m_2 x - b_2). \quad (6)$$

Now if these lines are to be AP', BP'' &c. of the figure, we must have their intercepts on Y equal respectively to $\frac{-c^2}{b_1}, \frac{-c^2}{b_2}$ and $\frac{c^2}{b_3}$; thus from (4) we shall have

$$\frac{b_2 - k_1 b_3}{1 - k_1} = \frac{-c^2}{b_1}, \text{ whence } k_1 = \frac{c^2 + b_1 b_2}{c^2 + b_3 b_1}.$$

To obtain the lines AQ', BQ'' &c., we should have to make the intercepts equal to $\frac{c^2}{b_1}, \frac{c^2}{b_2}$ and $\frac{c^2}{b_3}$ which will give $k_1 = \frac{c^2 - b_1 b_2}{c^2 - b_1 b_3}$. If we write



$$k_1 = \frac{\pm c^2 - b_1 b_2}{\pm c^2 - b_3 b_1}$$

we shall cover both cases at once. Substituting this value of k_1 and values of k_2 and k_3 similarly obtained, in (4), (5) and (6), we have

$$[\pm c^2(m_1 - m_2) + b_3(m_2 b_1 - m_1 b_2)]x - b_3(b_1 - b_2)y \pm c^2(b_1 - b_2) = 0, \quad (7)$$

$$[\pm c^2(m_2 - m_3) + b_1(m_3 b_2 - m_2 b_3)]x - b_1(b_2 - b_3)y \pm c^2(b_2 - b_3) = 0, \quad (8)$$

$$[\pm c^2(m_3 - m_1) + b_2(m_1 b_3 - m_3 b_1)]x - b_2(b_3 - b_1)y \pm c^2(b_3 - b_1) = 0. \quad (9)$$

If (7), (8) and (9) be added they vanish identically, showing that the three lines represented by them pass through a single point. Thus the proposition is proved.

The following is a special case of this proposition.

Let the triangle ABC be inscribed in a conic: take any point P as a pole, and find its polar, say L : from the points P_1 , P_2 , P_3 , where the sides of ABC cut L , draw lines through P , and find their poles, which will of course fall on L : call these poles P' , P'' and P''' : then $P'A$, $P''B$ and $P'''C$ meet in a point.

To prove this, draw the diameter through P , cutting L in O ; then we have only to show that $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' =$ a constant.

L is parallel to the diameter conjugate to OP ; referring the conic to this pair of conjugate diameters, let its equation be

$$\frac{x^2}{a_1^2} + \frac{y^2}{\pm b_1^2} = 1; \quad (10)$$

then that of its polar for the point (x_1, y_1) is

$$\frac{xx_1}{a_1^2} + \frac{yy_1}{\pm b_1^2} = 1. \quad (11)$$

For the point P , $y_1 = 0$ and $x_1 = a$, say; then

$$x = \frac{a_1^2}{a}. \quad (12)$$

Take any line through P , as

$$y = m(x - a). \quad (13)$$

To find the pole of this line compare (13) with (11), the conditions that they shall represent the same line are

$$m = -\frac{\pm b_1^2 x_1}{a_1^2 y_1} \text{ and } -ma = \frac{\pm b_1^2}{y_1}.$$

These conditions give

$$x_1 = \frac{a_1^2}{a}, \quad y_1 = \frac{\pm b_1^2}{-ma}.$$

Now let us find the portion of L intercepted between the line of (13) and OP ; \therefore in (13) let $x = a_1^2 \div \alpha$, and call the intercept y_2 , therefore

$$y_2 = m \left(\frac{a_1^2}{\alpha} - \alpha \right) = m \cdot \frac{a_1^2 - \alpha^2}{\alpha}.$$

Multiply together the values of y_1 and y_2 , therefore

$$y_1 y_2 = -\frac{\pm b_1^2}{\alpha^2} (a_1^2 - \alpha^2), \text{ a constant.}$$

Hence $OP_1 \times OP' = OP_2 \times OP'' = OP_3 \times OP''' =$ a constant. This proves the proposition for the hyperbola and ellipse, and it may likewise be easily proved for the parabola. If P be a focus, of course L is a directrix.

If we eliminate c between any two of equations (7), (8), (9), we shall find the equation of the locus of O' and O'' when c varies. Solving for $\pm c^2$, and placing for brevity $b_1 - b_2 = \beta_3$, $b_2 - b_3 = \beta_1$, $b_3 - b_1 = \beta_2$, $m_1 - m_2 = \mu_3$, &c., and $m_2 b_1 - m_1 b_2 = \delta_3$ &c., we have

$$\pm c^2 = \frac{b_3(\beta_3 y - \delta_3 x)}{\beta_3 + \mu_3 x} = \frac{b_1(\beta_1 y - \delta_1 x)}{\beta_1 + \mu_1 x} = \frac{b_2(\beta_2 y - \delta_2 x)}{\beta_2 + \mu_2 x}.$$

Taking either pair of these values of $\pm c^2$ we shall arrive at the same result after expansion and collection of terms, viz.;

$$(m_1 b_1 \delta_1 + m_2 b_2 \delta_2 + m_3 b_3 \delta_3) x^2 - (\mu_1 b_2 b_3 + \mu_2 b_3 b_1 + \mu_3 b_1 b_2) x y + (b_1^2 \delta_1 + b_2^2 \delta_2 + b_3^2 \delta_3) x + \beta_1 \beta_2 \beta_3 y = 0. \quad (14)$$

This equation is perfectly symmetrical with respect to the three sides of the triangle, as it should be; and it represents a hyperbola. If we substitute A , B , C and D for the coefficients, so that the equation becomes

$$Ax^2 + Bxy + Cx + Dy = 0,$$

we find for the equations of the asymptotes

$$x = -\frac{D}{B}, \quad y = -\frac{A}{B}x + \frac{AD - BC}{B^2},$$

hence the coordinates of the center are

$$x = -\frac{D}{B} \text{ and } y = \frac{2AD - BC}{B^2}.$$

If the co-ordinates of the vertices of the triangle ABC which are respectively

$$x_1 = -\frac{\beta_1}{\mu_1}, \quad y_1 = -\frac{\delta_1}{\mu_1}, \quad x_2 = -\frac{\beta_2}{\mu_2}, \quad y_2 = -\frac{\delta_2}{\mu_2}, \text{ etc.,}$$

be substituted in the equation of the hyperbola, they will each be found to satisfy it; hence the hyperbola passes through the vertices of the triangle.